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LOW MEDIAN AND LEAST ABSOLUTE RESIDUAL ANALYSIS OF TWO-WAY TABL--ETC(U)

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by

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LOW MEDIAN AND LEAST ABSOLUTE RESIDUAL ANALYSIS

OF TWO-WAY TABLES

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ABSTRACT

Some properties of and extensions to Tukey's method of median polish, an exploratory robust additive decomposition of a two-way table, are presented using the low median. If the table entries are rational numbers, then this modified iteration process must stop after a finite number of steps. However, even for tables of bounded dimension the number of iterations can be arbitrarily large. For the special case of 3 by 3 tables, the sum of absolute residuals is often (but not always) minimized by median polish, especially for tables with strong row or column effects. Methods designed to supplement the polishing process by increasing the number of zero residuals and to obtain a least absolute residual solution are developed.

SOME KEY WORDS: Median polish, robustness, resistance, exploratory data analysis, L1 estimation.

1. INTRODUCTION AND SUMMARY

Exploratory data analytic methods are designed to be practical tools for the identification of useful structures and features of complicated assemblages of recorded information. Although most of these methods seem empirically to work quite well in practice with actual data, often their exact theoretical properties have not yet been worked out in detail.

We will focus on the method of median polish, a robust and resistant exploratory method for computing an additive decomposition of a two-way table proposed by Tukey (1970, 1977). This method has also been considered by McNeil (1977), Mosteller & Tukey (1977), and Velleman & Hoaglin (1981). We will provide results concerning the convergence properties of this iterative procedure, and will shed some light on the relationship between median polish and least absolute residual solutions, which have been considered for two-way tables by Armstrong & Frome (1976, 1979).

The low median of a sample is defined to be the median if the sample size is an odd number, and the lower of the middle two values if the sample size is even. The low median minimizes the sum of absolute residuals just as the median itself does, and has the potential advantage of assuring the presence of at least one zero residual. We will work exclusively with low medians here; this simplifies some results by rendering the case of even sample size nearly as tractable as the odd case. The use of high medians in place of low medians would, of course, provide equivalent properties.

An additive decomposition of a two-way table, say of the r by c matrix Z, is a vector x of row effects, a vector y of column effects, and a matrix R of residuals such that

$$Z(i,j) = x(i) + y(j) + R(i,j), \text{ all } i,j. \tag{1.1}$$

Whenever (1.1) holds for some vectors x and y, we will say that the tables Z and R are additively equivalent. A decomposition can be chosen by specifying the row and column effects, which then determine the residuals. A good fit results in small residuals and a perfectly additive table, $x(i) + y(j)$, that is close to the original data table.

Median polish can begin with either rows or columns. Working alternately with rows and columns, medians are subtracted from the current table entries and added to the corresponding effects. One such step (either rows or columns) is called a "half iteration" or a "half-step." This process is repeated until the median of each row and each column is zero. The low median polish considered here is this same procedure, but with the low median used in place of the median.

The method of median polish was introduced by Tukey (1970, Chapter 16, page 35) as a procedure to be iterated until all rows and columns have median zero, although it is believed that termination after several steps will usually suffice. In Section 2 we will prove that the iteration process must stop if the table entries are rational numbers, but can take arbitrarily long even for integer tables of a given fixed size. The 3 by 3 case is considered in Section 3, and it is proven that the median polish solution often (although not always) minimizes the sum of absolute residuals (the L1 norm). Fortunately, this minimization tends to happen for a class of tables that are especially likely to arise in practice, namely those with strong row or column effects.

Section 4 considers two procedures that can supplement a median polish: finding a nearby solution that maximizes the number of zero residuals, and obtaining a nearby least absolute residual solution if this is not already achieved. Although a least absolute residual procedure can be done independently of median polish, for example by linear programming or other methods (Bloomfield & Steiger, 1980), there may be some advantage to staying near to the median polish solution. Because of nonuniqueness of least absolute residual solutions, an answer near to the original data of the table may well be preferable.

2. CONVERGENCE OF LOW MEDIAN POLISH.

Empirically, the process of median polish seems to iterate for several steps, with smaller medians being subtracted at each stage, then finally stop because the median of each row and column becomes zero. Thus it appears in practice that the iteration process actually stops after a finite number of steps. We will see that although it can be proven that this is the case for tables with rational entries, the process can take a very long time.

THEOREM 2.1: The low median analysis of a two way table with rational entries will stop iterating after a finite number of iterations. Convergence will be achieved and cycling is impossible.

PROOF: Without loss of generality, we may assume that the entries of the table are integers because we can multiply each entry by the least common multiple of the denominators of all table entries. This common scale factor will not affect the basic process of median polish and will change neither the number of, nor the nature of, the iterations. For a table of integers the low median must itself be an integer, so it follows that the sum of absolute residuals can change only in integral steps and can therefore decrease for only a finite number of iterations. The lemma that follows will complete the proof by showing that the only way the table can change while the sum of absolute residuals stays constant is if the number of strictly negative residuals decreases. This number is therefore an upper bound on the number of further iterations before the process stops. The lemma is stated for a half-iteration on rows but it applies, without loss of generality, with the words "rows" and "columns" interchanged.

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LEMMA 2.1. If after a half-iteration of low median polish on rows, the sum of absolute residuals in a table of residuals remains unchanged while some of the residuals do change, then

(a) there must be an even number of columns

and (b) the number of strictly negative residuals must decrease.

PROOF. Clearly the sum of absolute residuals in each row must remain unchanged because these cannot increase when a low median is subtracted. Because subtracting a nonzero median from an odd number of values in a given row must reduce the sum of absolute residuals, if the number of columns were odd there could be no change in the table, establishing result (a). Because subtraction of a positive low median from a group of numbers must decrease the sum of absolute values, all low medians must be nonpositive. Because the table does change, the low medians cannot all be zero. Therefore the low median of at least one row is negative. Subtracting such a negative value cannot change a zero or positive entry into a negative one, but it will change the low median entry itself into a new zero residual. This will eliminate at least one negative residual from the table.

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There is no simple upper bound on the number of iterations required for all tables of a given dimension. In fact there is no upper bound at all as is shown by the following counterexample. This will also suggest that median polish lacks a certain kind of robustness, because a small change in only one entry of a table can have far reaching consequences. Bear in mind that this is an extreme pathological case and is not typical behavior of an otherwise generally robust and resistant procedure.

Begin with the following fairly additive five by five table. Because the dimensions of row and column are both odd, low median polish is equivalent to median polish (as well as to high median polish) in this example.

4	3	2	2	1	
5	4	3	3	2	
6	5	4	3	2	(2.1)
6	5	5	4	3	
7	6	6	5	4	

After two half-steps of median polish starting with rows, we converge to a table with zero row and column medians:

0	0	0	1	1	
0	0	0	1	1	
0	0	0	0	0	(2.2)
-1	-1	0	0	0	
-1	-1	0	0	0	

If, however, we replace the value 4 in the upper left corner of (2.1), with the small perturbation 4+e instead, then after two half-steps we obtain a slightly different result:

$$\begin{array}{rrrrr} e & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 & 0 \end{array} \qquad (2.3)$$

Although this does not appear very different from the unperturbed result (2.2), note that the median of the first row is now e and is not zero. In fact, we have entered a long spiral cycling process which will require approximately 4/e half-steps before converging. Note that the smaller the perturbation, the longer the iteration process. If e is the reciprocal of an integer, then the final table of residuals will be

$$\begin{array}{rrrrr} e & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \qquad (2.4)$$

Although this table of residuals looks very different than the unperturbed result (2.2), if we replace e by zero, then they are additively equivalent. Moreover, the sum of absolute residuals has been nontrivially reduced from 8 to 4.

During the iteration process from (2.3) to (2.4) the reduction in the sum of absolute residuals was the same at each step. This suggests that in such pathological cases stopping early can result in a substantial and unnecessarily large sum of absolute residuals. For example, if e=.0001, then 40000 half-steps are required for congergence. If only 10 of these are performed, then the reduction in sum of absolute residuals is only from 8.0001 to 7.9991 instead of reaching the lower limit of 4.0001.

It is an open question whether there exists a table (necessarily containing some irrational entries) which never stops iterating because the sum of absolute residuals decreases indefinitely towards a limit. The answer, however, is not crucial because we know that even integer tables can take arbitrarily long to converge.

3. MEDIAN POLISH IS OFTEN OPTIMAL FOR 3 BY 3 TABLES.

Although median polish need not always yield an L1 solution (one that minimizes the sum of absolute residuals), for the special case of 3 by 3 tables it often will. Fortunately, this happens in precisely those tables likely to arise in practice, namely those with strong row or column effects.

THEOREM 3.1: If the first half-step of median polish in a three by three table results in a row or a column of zeroes, then median polish will converge in at most three steps to an L1 solution.

Details of the proof will be omitted. It proceeds by exhaustion of all possible cases, of which there are only finitely many, and relies on the following lemma to recognize when an L1 solution has been reached.

LEMMA 3.1: Every three by three table can be put into one of the following two forms by permuting rows and columns, and by subtracting row and column effects. Signs (+ and -) denote numbers either of that sign or zero. Any table in one of these two forms minimizes the sum of absolute residuals over all additively equivalent tables.

$$\begin{matrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{matrix} \qquad\qquad \begin{matrix} 0 & 0 & 0 \\ 0 & + & - \\ 0 & - & + \end{matrix} \tag{3.1}$$

PROOF: The L1 property can be established from the fact that if there were an additively equivalent table with smaller L1 norm, then by convexity every table that is a convex linear combination of the two must also reduce the L1 norm. A contradiction can be reached by moving only slightly from the original table in this family of tables so that no nonzero element changes its sign. It can then be verified by cancellation of terms that the L1 norm cannot decrease.

For example, this one-parameter of tables additively equivalent to a table of the first type in (3.1) can be represented as

$$\begin{bmatrix} 0 & Z(1,2) & -Z(1,3) \\ -Z(2,1) & 0 & Z(2,3) \\ Z(3,1) & -Z(3,2) & 0 \end{bmatrix} + t \begin{bmatrix} x(1)+y(1) & x(1)+y(2) & x(1)+y(3) \\ x(2)+y(1) & x(2)+y(2) & x(2)+y(3) \\ x(3)+y(1) & x(3)+y(2) & x(3)+y(3) \end{bmatrix} \quad (3.2)$$

where the $Z(i,j)$ are all nonnegative. By choosing a positive value for t small enough so that $t|x(i)+y(j)| < Z(i,j)$ whenever $Z(i,j)$ is nonzero, we find for representative terms that

$$| Z(1,2) + tx(1) + ty(1)| \geq | Z(1,2)| + tx(1) + ty(1)$$

and (3.3)

$$|-Z(1,3) + tx(1) + ty(3)| \geq |-Z(1,3)| - tx(1) - ty(3)$$

Doing this with the other off-diagonal terms, summing all the terms, and cancelling, we find that the L1 norm of (3.2) is greater than or equal to the L1 norm of the original table (on the left in (3.2)). The proof for the second form in (3.1) proceeds similarly.

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Here is an illustration of the known result that median polish need not result in an L1 solution:

$$\begin{matrix} 1 & 6 & 3 \\ 5 & 9 & 2 \\ 6 & 4 & 7 \end{matrix} \tag{3.4}$$

Starting with columns yields an L1 norm of 14; beginning with rows yields a smaller L1 norm of 12. However, the L1 norm can be reduced to 11 using row effects (0, 3, 4) and column effects (2, 6, 3), a solution unattainable by median polish alone. The residuals for these three solutions, respectively, are

$$\begin{matrix} -4 & 0 & 0 \\ 0 & -3 & -1 \\ 0 & -3 & -3 \end{matrix} \qquad \begin{matrix} -2 & 0 & 0 \\ 0 & 1 & -3 \\ 0 & -5 & 1 \end{matrix} \qquad \begin{matrix} -1 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & -6 & 0 \end{matrix} \tag{3.5}$$

4. FORCING ZERO RESIDUALS AND OBTAINING AN L1 FIT.

Upon completion of low median polish of an r by c table, each row and each column will contain at least one zero residual, and there will therefore be at least $\max(r,c)$ zeroes in the table of residuals. This generally falls short of the number of zeroes attainable by some linear fit that attains median polished form. We will show that there exists a set of effects such that the residual table is in low median polished form (i.e. the low median of each row and column is zero) and there are at least $r + c - 1$ zero residuals. As a result, we will be left with no more non-zero residuals than there are residual degrees of freedom. This may be an advantage in the further examination of the residuals. Following this, a least absolute deviation fit can be obtained in a straightforward way.

THEOREM 4.1: For every r by c table, there exists a set of effects such that the residual table has at least $r + c - 1$ zero values and has zero low medians for each row and each column.

PROOF: Begin with the low median polish procedure. If there are fewer than $r + c - 1$ zeroes, then by a dimensionality argument there exists an additive perturbation of the residual table that increases the number of zeroes by one without changing the low medians of the rows and columns. This perturbation table can be constructed, for example, using a graph with $r + c$ vertices, one for each row and one for each column. Two vertices will be connected by an edge if and only if one is a row vertex, the other is a column vertex, and there is a zero in the table at that row and column. This graph cannot be connected because a connected graph on n vertices must have at least $n-1$ edges. Divide the vertices into two nonempty disconnected sets. Define effects to be zero for the rows and columns in the first set, t for the rows in the second set, and $-t$ for the columns in the second set. Regardless of the value of t, when this perturbation table is added to the polished table the zeroes will be preserved and only the nonzero entries will be changed; for small t no signs will be changed and therefore the low median conditions will be unaffected. Choose t so that one nonzero entry becomes zero while no others change sign. If possible, the sign of t should be chosen so that the sum of absolute residuals decreases, although this is not a necessary step of the proof. By repeating this process on the resulting table, increasing the number of zero residuals by one each time, the result follows.

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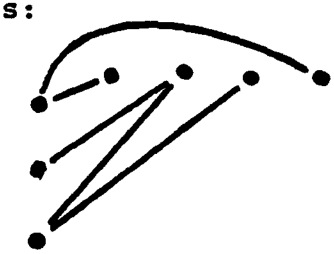
As an example of this connected graph argument, consider a polished table together with the graph connecting row vertices to column vertices where there are zeroes:

0	3	-1	0
-2	0	2	1
5	0	0	-1



(4.1)

Placing zeroes as effects for one connected component of the graph (row 1, column 1, and column 4) and choosing $t=1$ for the second component we obtain the additive table

$$\begin{array}{c|cccc} & 0 & -1 & -1 & 0 \\ \hline 0 & 0 & -1 & -1 & 0 \\ 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{array} \tag{4.2}$$

When this is added to our original table, left side of (4.1), a new sixth zero residual is introduced:

$$\begin{array}{cccc} 0 & 2 & -2 & 0 \\ -1 & 0 & 2 & 2 \\ 6 & 0 & 0 & 0 \end{array} \tag{4.3}$$

The low median polish has been preserved from (4.1) to (4.3) because no signs have been allowed to change.

After following these steps, an L1 fit can be obtained using a method related to that of Bloomfield and Steiger (1980). We will assume that our table is nondegenerate, i.e. it has no solution with more than $r + c - 1$ zero residual values. By adding small independent continuous random variables to the entries of a degenerate table, with probability one we can obtain a nondegenerate table that is close to the given table.

An L1 solution might be obtained as follows. For each of the $r + c - 1$ locations of zeroes in the table of residuals, find the one-dimensional space of additive perturbation tables (each of whose entries, by the connected graph construction above, may be assumed to be either 0, t, or $-t$) having zeroes at the other $r + c - 2$ locations. It may be that small values of t (either positive or negative) can decrease the sum of absolute residuals when the original table is added to the perturbation table. If this sum can be reduced, then increase the magnitude of t, while preserving its sign, until the minimum sum is attained; this might be computed by using each nonzero absolute value of the original table as a candidate for the magnitude of t. After this, the row and column low medians are no longer necessarily zero, and it may be necessary to repeat the processes of low median polish and the forcing of zeroes. On the other hand, if the sum of absolute residuals cannot be reduced in this way for any of the zeroes, then the following lemma shows that the current fit has minimized the sum of absolute residuals over all possible additive fits.

LEMMA 4.1. Let Z denote a nondegenerate r by c table with $r + c - 1$ zeroes. If the L1 norm of Z cannot be decreased by considering only those additively equivalent residual tables with zeroes at $r + c - 2$ (i.e. all but one) of the same places, then the L1 norm of Z is a minimum over ALL additively equivalent tables.

PROOF. Let $\|\cdot\|$ denote the L1 norm. We will proceed indirectly by assuming that there does exist an additive perturbation table T satisfying $\| Z + T \| < \| Z \|$, and showing that this assumption leads to a contradiction. Begin by constructing tables $U^{(1)},\ldots,U^{(r+c-1)}$ where each table entry is 0, 1, or -1; each $U^{(k)}$ has a zero at every zero of Z except the k^{th} one; yet no $U^{(k)}$ is identically zero. Such $U^{(k)}$ can be constructed, for example, using the connected graph argument of the proof of Theorem 4.1.

The set of tables $\{U^{(1)},\ldots,U^{(r+c-1)}\}$ is linearly independent because at the location corresponding to the kth zero of Z, only $U^{(k)}$ is nonzero. Because the dimensionality is correct, $\{U^{(1)},\ldots,U^{(r+c-1)}\}$ is a basis for the vector space of all additive r by c tables. Therefore there exist coefficients a_k such that

$$T = \sum_{k=1}^{r+c-1} a_k U^{(k)} \tag{4.4}$$

We will also need an e in (0,1) satisfying

$$e \sum_{k=1}^{r+c-1} |a_k U_{ij}^{(k)}| < |Z_{ij}| \qquad \text{whenever} \quad Z_{ij} \neq 0 \tag{4.5}$$

Because the number of constraints is finite, such an e does exist.

The following computations, from (4.6) to (4.9) will show that the change in norm from Z to Z + eT is equal to the sum over k of the change in norm from Z to $Z + ea_kU^{(k)}$. Begin by breaking the sum of absolute residuals into two parts:

$$\|Z + eT\| - \|Z\| = \tag{4.6}$$
$$\sum_{Z_{ij}=0} e \left| \sum_k a_k U_{ij}^{(k)} \right| + \sum_{Z_{ij}\neq 0} \left(\left| Z_{ij} + e \sum_k a_k U_{ij}^{(k)} \right| - |Z_{ij}| \right)$$

The first summation in (4.6) can be simplified because for a given (i,j) pair there is exactly one value of k with nonzero $U_{ij}^{(k)}$; moreover this $|U_{ij}^{(k)}| = 1$. The second summation in (4.6) can be simplified using the fact that $|a| > |b|$ implies $|a + b| - |a| = [\text{sign}(a)]\, b$; this is why a small e was needed. Thus (4.6) becomes

$$e \sum_k |a_k| + e \sum_{Z_{ij}\neq 0} \text{sign}(Z_{ij}) \sum_k a_k U_{ij}^{(k)} \tag{4.7}$$

Exchanging the order of summation in the second term, then applying the above arguments in reverse, (4.7) becomes

$$\sum_k \left\{ \sum_{Z_{ij}=0} e\, |a_k U_{ij}^{(k)}| + \sum_{Z_{ij}\neq 0} \left(|Z_{ij} + e a_k U_{ij}^{(k)}| - |Z_{ij}| \right) \right\} \tag{4.8}$$

which we recognize to be

$$\sum_k \left(\|Z + e a_k U^{(k)}\| - \|Z\| \right) \tag{4.9}$$

which must be nonnegative because for each k the table $e a_k U^{(k)}$ has $r + c - 2$ zeroes that coincide with zeroes of Z and hence, by a hypothesis of the lemma, each term in the sum of (4.6) is nonnegative.

The net result of (4.6) through (4.9) is therefore

$$\|Z + eT\| \geq \|Z\| \tag{4.10}$$

But $\|Z + tT\|$ is the sum of convex functions $|Z_{ij} - tT_{ij}|$ of t, and is therefore a convex function of t. Thus

$$\|Z + eT\| < \|Z\| \tag{4.11}$$

follows from the assumption that $\|Z + T\| < \|Z\|$ and the fact that e is between 0 and 1. The contradiction of (4.10) and (4.11) completes the proof. ||

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